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# MECHANICAL DESIGN FUNDAMENTALS

# CHAPTER 1
# CLASSICAL MECHANICS

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The aim of this chapter is to present the concepts and results of newtonian dynamics which are required in a discussion of rigid-body motion. The detailed analysis of particular rigid-body motions is not included. The chapter contains a few topics which, while not directly needed in the discussion, either serve to round out the presentation or are required elsewhere in this handbook.

## 1.1 INTRODUCTION

The study of classical dynamics is founded on Newton's three laws of motion and on the accompanying assumptions of the existence of absolute space and absolute time. In addition, in problems in which gravitational effects are of importance, Newton's law of gravitation is adopted. The objective of the study is to enable one to predict, given the initial conditions and the forces which act, the evolution in time of a mechanical system or, given the motion, to determine the forces which produce it.

The mathematical formulation and development of the subject can be approached in two ways. The vectorial method, that used by Newton, emphasizes the vector quantities force and acceleration. The analytical method, which is largely due to Lagrange, utilizes the scalar quantities work and energy. The former method is the more physical and generally possesses the advantage in situations in which dissipative forces are present. The latter is more mathematical and accordingly is very useful in developing powerful general results.

## 1.2 THE BASIC LAWS OF DYNAMICS

The "first law of motion" states that a body which is under the action of no force remains at rest or continues in uniform motion in a straight line. This statement is also

known as the "law of inertia," inertia being that property of a body which demands that a force is necessary to change its motion. "Inertial mass" is the numerical measure of inertia. The conditions under which an experimental proof of this law could be carried out are clearly not attainable.

In order to investigate the motion of a system it is necessary to choose a frame of reference, assumed to be rigid, relative to which the displacement, velocity, etc., of the system are to be measured. The law of inertia immediately classifies the possible frames of reference into two types. For, suppose that in a certain frame $S$ the law is found to be true; then it must also be true in any frame which has a constant velocity vector relative to $S$. However, the law is found not to be true in any frame which is in accelerated motion relative to $S$. A frame of reference in which the law of inertia is valid is called an "inertial frame," and any frame in accelerated motion relative to it is said to be "noninertial." Any one of the infinity of inertial frames can claim to be at rest while all others are in motion relative to it. Hence it is not possible to distinguish, by observation, between a state of rest and one of uniform motion in a straight line. The transformation rules by which the observations relative to two inertial frames are correlated can be deduced from the second law of motion.

Newton's "second law of motion" states that in an inertial frame the force acting on a mass is equal to the time rate of change of its linear momentum. "Linear momentum," a vector, is defined to be the product of the inertial mass and the velocity. The law can be expressed in the form

$$d/dt(m\mathbf{v}) = \mathbf{F} \tag{1.1}$$

which, in the many cases in which the mass $m$ is constant, reduces to

$$m\mathbf{a} = \mathbf{F} \tag{1.2}$$

where $a$ is the acceleration of the mass.

The "third law of motion," the "law of action and reaction," states that the force with which a mass $m_i$ acts on a mass $m_j$ is equal in magnitude and opposite in direction to the force which $m_j$ exerts on $m_i$. The additional assumption that these forces are collinear is needed in some applications, e.g., in the development of the equations governing the motion of a rigid body.

The "law of gravitation" asserts that the force of attraction between two point masses is proportional to the product of the masses and inversely proportional to the square of the distance between them. The masses involved in this formula are the gravitational masses. The fact that falling bodies possess identical accelerations leads, in conjunction with Eq. (1.2), to the proportionality of the inertial mass of a body to its gravitational mass. The results of very precise experiments by Eotvös and others show that inertial mass is, in fact, equal to gravitational mass. In the future the word mass will be used without either qualifying adjective.

If a mass in motion possesses the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the origins of two inertial frames $S_1$ and $S_2$, respectively, and if further $S_1$ and $S_2$ have a relative velocity $\mathbf{V}$, then it follows from Eq. (1.2) that

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_2 + \mathbf{V}t_2 + \text{const} \\ t_1 &= t_2 + \text{const} \end{aligned} \tag{1.3}$$

in which $t_1$ and $t_2$ are the times measured in $S_1$ and $S_2$. The transformation rules Eq. (1.3), in which the constants depend merely upon the choice of origin, are called "galilean transformations." It is clear that acceleration is an invariant under such transformations.

The rules of transformation between an inertial frame and a noninertial frame are considerably more complicated than Eq. (1.3). Their derivation is facilitated by the application of the following theorem: a frame $S_1$ possesses relative to a frame $S$ an angular velocity $\boldsymbol{\omega}$ passing through the common origin of the two frames. The time rate of change

of any vector **A** as measured in $S$ is related to that measured in $S_1$ by the formula

$$(d\mathbf{A}/dt)_S = (d\mathbf{A}/dt)_{S_1} + \boldsymbol{\omega} \times \mathbf{A} \tag{1.4}$$

The interpretation of Eq. (1.4) is clear. The first term on the right-hand side accounts for the change in the magnitude of **A**, while the second corresponds to its change in direction.

If $S$ is an inertial frame and $S_1$ is a frame rotating relative to it, as explained in the statement of the theorem, $S_1$ being therefore noninertial, the substitution of the position vector **r** for **A** in Eq. (1.4) produces the result

$$\mathbf{v}_{abs} = \mathbf{v}_{rel} + \boldsymbol{\omega} \times \mathbf{r} \tag{1.5}$$

In Eq. (1.5) $\mathbf{v}_{abs}$ represents the velocity measured relative to $S$, $\mathbf{v}_{rel}$ the velocity relative to $S_1$, and $\boldsymbol{\omega} \times \mathbf{r}$ is the transport velocity of a point rigidly attached to $S_1$. The law of transformation of acceleration is found on a second application of Eq. (1.4), in which **A** is replaced by $\mathbf{v}_{abs}$. The result of this substitution leads directly to

$$(d^2\mathbf{r}/dt^2)_S = (d^2\mathbf{r}/dt^2)_{S_1} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_{rel} \tag{1.6}$$

in which $\dot{\boldsymbol{\omega}}$ is the time derivative, in either frame, of $\boldsymbol{\omega}$. The physical interpretation of Eq. (1.6) can be shown in the form

$$\mathbf{a}_{abs} = \mathbf{a}_{rel} + \mathbf{a}_{trans} + \mathbf{a}_{cor} \tag{1.7}$$

where $\mathbf{a}_{cor}$ represents the Coriolis acceleration $2\boldsymbol{\omega} \times \mathbf{v}_{rel}$. The results, Eqs. (1.5) and (1.7), constitute the rules of transformation between an inertial and a noninertial frame. Equation (1.7) shows in addition that in a noninertial frame the second law of motion takes the form

$$m\mathbf{a}_{rel} = \mathbf{F}_{abs} - m\mathbf{a}_{cor} - m\mathbf{a}_{trans} \tag{1.8}$$

The modifications required in the above formulas are easily made for the case in which $S_1$ is translating as well as rotating relative to $S$. For, if $\mathbf{D}(t)$ is the position vector of the origin of the $S_1$ frame relative to that of $S$, Eq. (1.5) is replaced by

$$\mathbf{V}_{abs} = (d\mathbf{D}/dt)_S + \mathbf{v}_{rel} + \boldsymbol{\omega} \times \mathbf{r}$$

and consequently, Eq. (1.7) is replaced by

$$\mathbf{a}_{abs} = (d^2\mathbf{D}/dt^2)_S + \mathbf{a}_{rel} + \mathbf{a}_{trans} + \mathbf{a}_{cor}$$

In practice the decision as to what constitutes an inertial frame of reference depends upon the accuracy sought in the contemplated analysis. In many cases a set of axes rigidly attached to the earth's surface is sufficient, even though such a frame is noninertial to the extent of its taking part in the daily rotation of the earth about its axis and also its yearly rotation about the sun. When more precise results are required, a set of axes fixed at the center of the earth may be used. Such a set of axes is subject only to the orbital motion of the earth. In still more demanding circumstances, an inertial frame is taken to be one whose orientation relative to the fixed stars is constant.

## 1.3 THE DYNAMICS OF A SYSTEM OF MASSES

The problem of locating a system in space involves the determination of a certain number of variables as functions of time. This basic number, which cannot be reduced without the imposition of constraints, is characteristic of the system and is known as

its number of degrees of freedom. A point mass free to move in space has three degrees of freedom. A system of two point masses free to move in space, but subject to the constraint that the distance between them remains constant, possesses five degrees of freedom. It is clear that the presence of constraints reduces the number of degrees of freedom of a system.

Three possibilities arise in the analysis of the motion-of-mass systems. First, the system may consist of a small number of masses and hence its number of degrees of freedom is small. Second, there may be a very large number of masses in the system, but the constraints which are imposed on it reduce the degrees of freedom to a small number; this happens in the case of a rigid body. Finally, it may be that the constraints acting on a system which contains a large number of masses do not provide an appreciable reduction in the number of degrees of freedom. This third case is treated in statistical mechanics, the degrees of freedom being reduced by statistical methods.

In the following paragraphs the fundamental results relating to the dynamics of mass systems are derived. The system is assumed to consist of $n$ constant masses $m_i (i = 1, 2, \ldots, n)$. The position vector of $m_i$, relative to the origin $O$ of an inertial frame, is denoted by $\mathbf{r}_i$. The force acting on $m_i$ is represented in the form

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_{j=1}^{n} \mathbf{F}_{ij} \tag{1.9}$$

in which $\mathbf{F}_i^e$ is the external force acting on $m_i$, $\mathbf{F}_{ij}$ is the force exerted on $m_i$ by $m_j$, and $\mathbf{F}_{ii}$ is zero.

### 1.3.1 The Motion of the Center of Mass

The motion of $m_i$ relative to the inertial frame is determined from the equation

$$\mathbf{F}_i^e + \sum_{j=1}^{n} \mathbf{F}_{ij} = m_i \frac{d\mathbf{v}_i}{dt} \tag{1.10}$$

On summing the $n$ equations of this type one finds

$$\mathbf{F}^e + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{F}_{ij} = \sum_{i=1}^{n} m_i \frac{d\mathbf{v}_i}{dt} \tag{1.11}$$

where $\mathbf{F}^e$ is the resultant of all the external forces which act on the system. But Newton's third law states that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

and hence the double sum in Eq. (1.11) vanishes. Further, the position vector $\mathbf{r}_c$ of the center of mass of the system relative to $O$ is defined by the relation

$$\mathcal{M}\mathbf{r}_c = \sum_{i=1}^{n} m_i \mathbf{r}_i \tag{1.12}$$

in which $\mathcal{M}$ denotes the total mass of the system. It follows from Eq. (1.12) that

$$\mathcal{M}\mathbf{v}_c = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{1.13}$$

and therefore from Eq. (1.11) that

$$\mathbf{F}^e = \mathcal{M}\, d^2\mathbf{r}_c/dt^2 \tag{1.14}$$

which proves the theorem: the center of mass moves as if the entire mass of the system were concentrated there and the resultant of the external forces acted there.

Two first integrals of Eq. (1.14) provide useful results [Eqs. (1.15) and (1.16):

$$\int_{t_1}^{t_2} \mathbf{F}^e \, dt = \mathcal{M}\mathbf{v}_c(t_2) - \mathcal{M}\mathbf{v}_c(t_1) \tag{1.15}$$

The integral on the left-hand side is called the "impulse" of the external force. Equation (1.15) shows that the change in linear momentum of the center of mass is equal to the impulse of the external force. This leads to the conservation-of-linear-momentum theorem: the linear momentum of the center of mass is constant if no resultant external force acts on the system or, in view of Eq. (1.13), the total linear momentum of the system is constant if no resultant external force acts:

$$\int_1^2 \mathbf{F}^e \cdot \mathbf{r}_c = \frac{1}{2}\mathcal{M}v_c^2\Big]_1^2 \tag{1.16}$$

which constitutes the work-energy theorem: the work done by the resultant external force acting at the center of mass is equal to the change in the kinetic energy of the center of mass.

In certain cases the external force $\mathbf{F}_i^e$ may be the gradient of a scalar quantity $V$ which is a function of position only. Then

$$\mathbf{F}^e = -\partial V/\partial \mathbf{r}_c$$

and Eq (1.16) takes the form

$$\left[\frac{1}{2}\mathcal{M}\mathbf{v}_c^2 + V\right]_1^2 = 0 \tag{1.17}$$

If such a function $V$ exists, the force field is said to be conservative and Eq. (1.17) provides the conservation-of-energy theorem.

### 1.3.2 The Kinetic Energy of a System

The total kinetic energy of a system is the sum of the kinetic energies of the individual masses. However, it is possible to cast this sum into a form which frequently makes the calculation of the kinetic energy less difficult. The total kinetic energy of the masses in their motion relative to $O$ is

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \mathbf{v}_i^2$$

but

$$\mathbf{r}_i = \mathbf{r}_c + \sigma_i$$

where $\boldsymbol{\sigma}_i$ is the position vector of $m_i$ relative to the system center of mass $C$ (see Fig. 1.1).

Hence

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_c^2 + \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_c \cdot \dot{\boldsymbol{\sigma}}_i + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\boldsymbol{\sigma}}_i^2$$

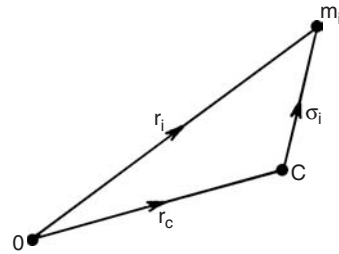


**FIG. 1.1**

but $$\sum_{i=1}^{n} m_i \boldsymbol{\sigma}_i = 0$$

by definition, and so

$$T = \frac{1}{2} \mathcal{M} \dot{\mathbf{r}}_c^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\boldsymbol{\sigma}}_i^2 \tag{1.18}$$

which proves the theorem: the total kinetic energy of a system is equal to the kinetic energy of the center of mass plus the kinetic energy of the motion relative to the center of mass.

### 1.3.3 Angular Momentum of a System (Moment of Momentum)

Each mass $m_i$ of the system has associated with it a linear momentum vector $m_i \mathbf{v}_i$. The moment of this momentum about the point $O$ is $\mathbf{r}_i \times m_i \mathbf{v}_i$. The moment of momentum of the motion of the system relative to $O$, about $O$, is

$$\mathbf{H}(O) = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i$$

It follows that

$$\frac{d}{dt} \mathbf{H}(O) = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \frac{d^2 \mathbf{r}_i}{dt^2}$$

which, by Eq. (1.10), is equivalent to

$$\frac{d}{dt} \mathbf{H}(O) = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^e + \sum_{i=1}^{n} \mathbf{r}_i \times \sum_{j=1}^{n} F_{ij} \tag{1.19}$$

It is now assumed that, in addition to the validity of Newton's third law, the force $F_{ij}$ is collinear with $\mathbf{F}_{ji}$ and acts along the line joining $m_i$ to $m_j$, i.e., the internal forces are central forces. Consequently, the double sum in Eq. (1.19) vanishes and

$$\frac{d}{dt} \mathbf{H}(O) = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^e = \mathbf{M}(O) \tag{1.20}$$

where $\mathbf{M}(O)$ represents the moment of the external forces about the point $O$. The following extension of this result to certain noninertial points is useful.

Let $A$ be an arbitrary point with position vector $\mathbf{a}$ relative to the inertial point $O$ (see Fig. 1.2). If $\boldsymbol{\rho}_i$ is the position vector of $m_i$ relative to $A$, then in the notation already developed

$$\mathbf{H}(A) = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \frac{d\mathbf{r}_i}{dt} = \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{a}) \times m_i \frac{d\mathbf{r}_i}{dt} = \mathbf{H}(O) - \mathbf{a} \times \mathcal{M} \mathbf{v}_c$$

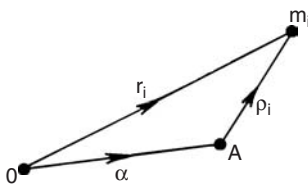


FIG. 1.2

Thus $(d/dt)\ \mathbf{H}(A) = (d/dt)\mathbf{H}(O) - \dot{\mathbf{a}} \times \mathcal{M}\mathbf{v}_c - \mathbf{a} \times \mathcal{M}(d\mathbf{v}_c/dt)$, which reduces on application of Eqs. (1.14) and (1.20) to

$$(d/dt)\mathbf{H}(A) = \mathbf{M}(A) - \dot{\mathbf{a}} \times \mathcal{M} v_c$$

The validity of the result

$$(d/dt)\mathbf{H}(A) = \mathbf{M}(A) \tag{1.21}$$

is assured if the point $A$ satisfies either of the conditions

1. $\dot{\mathbf{a}} = 0$; i.e., the point $A$ is fixed relative to $O$.
2. $\dot{\mathbf{a}}$ is parallel to $\mathbf{v}_c$; i.e., the point $A$ is moving parallel to the center of mass of the system.

A particular, and very useful case of condition 2 is that in which the point $A$ is the center of mass. The preceding results [Eqs. (1.20) and (1.21)] are contained in the theorem: the time rate of change of the moment of momentum about a point is equal to the moment of the external forces about that point if the point is inertial, is moving parallel to the center of mass, or is the center of mass.

As a corollary to the foregoing, one can state that the moment of momentum of a system about a point satisfying the conditions of the theorem is conserved if the moment of the external forces about that point is zero.

The moment of momentum about an arbitrary point $A$ of the motion relative to $A$ is

$$\mathbf{H}_{\text{rel}}(A) = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i \frac{d\boldsymbol{\rho}_i}{dt} = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{a}}) = H(A) + \dot{\mathbf{a}} \times \sum_{i=1}^{n} m_i \boldsymbol{\rho}_i \quad (1.22)$$

If the point $A$ is the center of mass $C$ of the system, Eq. (1.22) reduces to

$$\mathbf{H}_{\text{rel}}(C) = \mathbf{H}(C) \quad (1.23)$$

which frequently simplifies the calculation of $\mathbf{H}(C)$.

Additional general theorems of the type derived above are available in the literature. The present discussion is limited to the more commonly applicable results.

## 1.4 THE MOTION OF A RIGID BODY

As mentioned earlier, a rigid body is a dynamic system that, although it can be considered to consist of a very large number of point masses, possesses a small number of degrees of freedom. The rigidity constraint reduces the degrees of freedom to six in the most general case, which is that in which the body is translating and rotating in space. This can be seen as follows: The position of a rigid body in space is determined once the positions of three noncollinear points in it are known. These three points have nine coordinates, among which the rigidity constraint prescribes three relationships. Hence only six of the coordinates are independent. The same result can be obtained otherwise.

Rather than view the body as a system of point masses, it is convenient to consider it to have a mass density per unit volume. In this way the formulas developed in the analysis of the motion of mass systems continue to be applicable if the sums are replaced by integrals.

The six degrees of freedom demand six equations of motion for the determination of six variables. Three of these equations are provided by Eq. (1.14), which describes the motion of the center of mass, and the remaining three are found from moment-of-momentum considerations, e.g., Eq. (1.21). It is assumed, therefore, in what follows that the motion of the center of mass is known, and the discussion is limited to the rotational motion of the rigid body about its center of mass $C$.*

Let $\boldsymbol{\omega}$ be the angular velocity of the body. Then the moment of momentum about $C$ is, by Eq. (1.3),

$$\mathbf{H}(C) = \int_V \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\rho \, dV \quad (1.24)$$

*Rotational motion about any *fixed* point of the body is treated in a similar way.

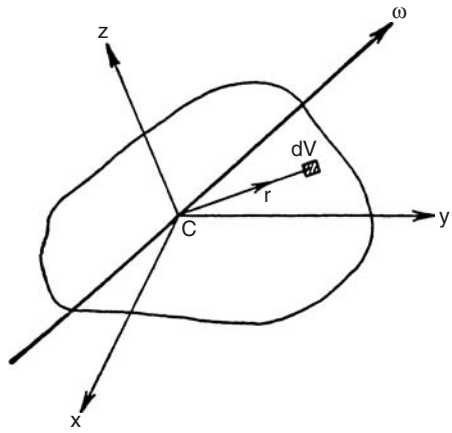


**FIG. 1.3**

where **r** is now the position vector of the element of volume $dV$ relative to $C$ (see Fig. 1.3), $\rho$ is the density of the body, and the integral is taken over the volume of the body. By a direct expansion one finds

$$\begin{aligned}\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) &= \mathbf{r}^2\boldsymbol{\omega} - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}) = \mathbf{r}^2\boldsymbol{\omega} - \mathbf{r}\mathbf{r} \cdot \boldsymbol{\omega} \\ &= \mathbf{r}^2\mathbf{I} \cdot \boldsymbol{\omega} - \mathbf{r}\mathbf{r} \cdot \boldsymbol{\omega} = (\mathbf{r}^2\mathbf{I} - \mathbf{r}\mathbf{r}) \cdot \boldsymbol{\omega}\end{aligned} \tag{1.25}$$

and hence

$$\mathbf{H}(C) = \boldsymbol{I}(C) \cdot \boldsymbol{\omega} \tag{1.25}$$

where

$$\boldsymbol{I}(C) = \int_V \rho(\mathbf{r}^2\mathbf{I} - \mathbf{r}\mathbf{r})\, dV \tag{1.26}$$

is the inertia tensor of the body about $C$.

In Eq. (1.26), **I** denotes the identity tensor. The inertia tensor can be evaluated once the value of $\rho$ and the shape of the body are prescribed. We now make a short digression to discuss the structure and properties of $\boldsymbol{I}(C)$.

For definiteness let $x$, $y$, and $z$ be an orthogonal set of cartesian axes with origin at $C$ (see Fig. 1.3). Then in matrix notation

$$\boldsymbol{I}(C) = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

where

$$I_{xx} = \int_V \rho(y^2 + z^2)\, dV$$

$$I_{xy} = \int_V \rho xy\, dV$$

. . . . . .

It is clear that:

1. The tensor is second-order symmetric with real elements.
2. The elements are the usual moments and products of inertia.

**3.** The moment of inertia about a line through $C$ defined by a unit vector $\mathbf{e}$ is

$$\mathbf{e} \cdot \boldsymbol{I}(C) \cdot \mathbf{e}$$

**4.** Because of the property expressed in condition 1, it is always possible to determine at $C$ a set of mutually perpendicular axes relative to which $\boldsymbol{I}(C)$ is diagonalized.

Returning to the analysis of the rotational motion, one sees that the inertia tensor $\boldsymbol{I}(C)$ is time-dependent unless it is referred to a set of axes which rotate with the body. For simplicity the set of axes $S_1$ which rotates with the body is chosen to be the orthogonal set in which $\boldsymbol{I}(C)$ is diagonalized. A space-fixed frame of reference with origin at $C$ is represented by $S$. Accordingly, from Eqs. (1.4) and (1.21),

$$[(d/dt)\mathbf{H}(C)]_S = [(d/dt)\mathbf{H}(C)]_{S_1} + \boldsymbol{\omega} \times \mathbf{H}(C) = \mathbf{M}(C) \tag{1.27}$$

which, by Eq. (1.25), reduces to

$$\boldsymbol{I}(C) \cdot (d\boldsymbol{\omega}/dt) + \boldsymbol{\omega} \times \boldsymbol{I}(C) \cdot \boldsymbol{\omega} = \mathbf{M}(C) \tag{1.28}$$

where

$$\mathbf{H}(C) = \mathbf{i}I_{xx}\omega_x + \mathbf{j}I_{yy}\omega_y + \mathbf{k}I_{zz}\omega_z \tag{1.29}$$

In Eq. (1.29) the $x$, $y$, and $z$ axes are those for which

$$\boldsymbol{I}(C) = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the conventional unit vectors. Equation (1.28) in scalar form supplies the three equations needed to determine the rotational motion of the body. These equations, the Euler equations, are

$$\begin{aligned} I_{xx}(d\omega_x/dt) + \omega_y\omega_z(I_{zz} - I_{yy}) &= M_x \\ I_{yy}(d\omega_y/dt) + \omega_z\omega_x(I_{xx} - I_{zz}) &= M_y \\ I_{zz}(d\omega_z/dt) + \omega_x\omega_y(I_{yy} - I_{xx}) &= M_z \end{aligned} \tag{1.30}$$

The analytical integration of the Euler equations in the general case defines a problem of classical difficulty. However, in special cases solutions can be found. The sources of the simplifications in these cases are the symmetry of the body and the absence of some components of the external moment. Since discussion of the various possibilities lies outside the scope of this chapter, reference is made to Refs. 1, 2, 6, and 7 and, for a survey of recent work, to Ref. 3. Of course, in situations in which energy or moment of momentum, or perhaps both, are conserved, first integrals of the motion can be written without employing the Euler equations. To do so it is convenient to have an expression for the kinetic energy $T$ of the rotating body. This expression is readily found in the following manner.

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\int_V \rho(\boldsymbol{\omega} \times \mathbf{r})^2\, dV \\ &= \frac{1}{2}\int_V \rho\boldsymbol{\omega} \cdot [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, dV \end{aligned}$$

which, by Eqs. (1.24), (1.25), and (1.26), is

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I}(C) \cdot \boldsymbol{\omega} \tag{1.31}$$

or, in matrix notation,

$$2T = (\omega_x \omega_y \omega_z) \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Equation (1.31) can be put in a simpler form by writing

$$T = \frac{1}{2} \omega^2 (\boldsymbol{\omega}/\omega) \cdot \boldsymbol{I}(C) \cdot (\boldsymbol{\omega}/\omega)$$

and hence

$$T = \frac{1}{2} I_{\omega\omega} \omega^2 \tag{1.32}$$

In Eq. (1.32) $I_{\omega\omega}$ is the moment of inertia of the body about the axis of the angular velocity vector $\boldsymbol{\omega}$.

## 1.5 ANALYTICAL DYNAMICS

The knowledge of the time dependence of the position vectors $\mathbf{r}_i(t)$ which locate an $n$-mass system relative to a frame of reference can be attained indirectly by determining the dependence upon time of some parameters $q_j$ ($j = 1, \ldots, m$) if the functional relationships

$$\mathbf{r}_i = \mathbf{r}_i(q_j, t) \qquad i = 1, \ldots, n; \quad j = 1, \ldots, m \tag{1.33}$$

are known. The parameters $q_j$ which completely determine the position of the system in space are called "generalized coordinates." Any $m$ quantities can be used as generalized coordinates on condition that they uniquely specify the positions of the masses. Frequently the $q_j$ are the coordinates of an appropriate curvilinear system.

It is convenient to define two types of mechanical systems:

1. A "holonomic system" is one for which the generalized coordinates and the time may be arbitrarily and independently varied without violating the constraints.
2. A "nonholonomic system" is such that the generalized coordinates and the time may not be arbitrarily and independently varied because of some (say $s$) nonintegrable constraints of the form

$$\sum_{i=1}^{m} A_{ji}\, dq_i + A_j\, dt = 0 \qquad j = 1, 2, \ldots, s \tag{1.34}$$

In the constraint equations [Eq. (1.34)] the $A_{ji}$ and $A_j$ represent functions of the $q_k$ and $t$. Holonomic and nonholonomic systems are further classified as "rheonomic" or "scleronomic," depending upon whether the time $t$ is explicitly present or absent, respectively, in the constraint equations.

### 1.5.1 Generalized Forces and d'Alembert's Principle

A virtual displacement of the system is denoted by the set of vectors $\delta\mathbf{r}_i$. The work done by the forces in this displacement is

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta\mathbf{r}_i \tag{1.35}$$

If the force $\mathbf{F}_i$, acting on the mass $m_i$, is separable in the sense that

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^c \tag{1.36}$$

in which the first term is the applied force and the second the force of constraint, then

$$\delta W = \sum_{i=1}^{n} (\mathbf{F}_i^a + \mathbf{F}_i^c) \left[ \sum_{j=1}^{m} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j + \frac{\partial \mathbf{r}_i}{\partial t} \delta t \right] \tag{1.37}$$

The generalized applied forces and the generalized forces of constraint are defined by

$$Q_j^a = \sum_{i=1}^{n} \mathbf{F}_i^a \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{1.38}$$

and

$$Q_j^c = \sum_{i=1}^{n} \mathbf{F}_i^c \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{1.39}$$

respectively. Hence, Eq. (1.37) assumes the form

$$\delta W = \sum_{j=1}^{m} Q_j^a \, \delta q_j + \sum_{j=1}^{m} Q_j^c \, \delta q_j + \sum_{i=1}^{n} (\mathbf{F}_i^a + \mathbf{F}_i^c) \cdot \frac{\partial \mathbf{r}_i}{\partial t} \delta t \tag{1.40}$$

If the virtual displacement is compatible with the instantaneous constraints $\delta t = 0$, and if in such a displacement the forces of constraint do work, e.g., if sliding friction is absent, then

$$\delta W = \sum_{j=1}^{m} Q_j^a \, \delta q_j \tag{1.41}$$

The assumption that a function $V(q_j, t)$ exists such that

$$Q_j^a = -\partial V / \partial q_j$$

leads to the result

$$\delta W = -\delta V \tag{1.42}$$

In Eq. (1.42), $V(q_j, t)$ is called the potential or work function.

The first step in the introduction of the kinetic energy of the system is taken by using d'Alembert's principle. The equations of motion [Eq. (1.10)] can be written as

$$\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i = 0$$

and consequently

$$\sum_{i=1}^{n} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0 \tag{1.43}$$

The principle embodied in Eq. (1.43) constitutes the extension of the principle of virtual work to dynamic systems and is named after d'Alembert. When attention is confined to $\delta \mathbf{r}_i$ which represent virtual displacements compatible with the instantaneous constraints and to forces $\mathbf{F}_i$ which satisfy Eqs. (1.36) and (1.41), the principle states that

$$\sum_{j=1}^{m} Q_j^a \, \delta q_j = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \tag{1.44}$$

### 1.5.2 The Lagrange Equations

The central equations of analytical mechanics can now be derived. These equations, which were developed by Lagrange, are presented here for the general case of a rheonomic nonholonomic system consisting of $n$ masses $m_i$, $m$ generalized coordinates $q_i$, and $s$ constraint equations

$$\sum_{j=1}^{m} A_{kj}\, dq_j + A_k\, dt = 0 \qquad k = 1, 2, \ldots, s \tag{1.45}$$

The equations are found by writing the acceleration terms in d'Alembert's principle [Eq. (1.43)] in terms of the kinetic energy $T$ and the generalized coordinates. By definition

$$T = \frac{1}{2} \sum_{1}^{n} m_i \dot{\mathbf{r}}_i^2$$

where

$$\dot{\mathbf{r}}_i = \sum_{j=1}^{m} \frac{\partial \mathbf{r}_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial \mathbf{r}_i}{dt} \qquad i = 1, 2, \ldots, n$$

Thus

$$\partial \dot{\mathbf{r}}_i / \partial \dot{q}_j = \partial \mathbf{r}_i / \partial q_j \qquad \partial \dot{\mathbf{r}}_i / \partial q_j = (d/dt)(\partial \mathbf{r}_i / \partial q_j)$$

$$\partial T / \partial q_j = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad \text{and} \qquad \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Accordingly,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \sum_{i=1}^{n} m_r \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad j = 1, 2, \ldots, m \tag{1.46}$$

and by summing over all values of $j$, one finds

$$\sum_{j=1}^{m} \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \delta q_j = \sum_{i=1}^{n} m_i\, \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \tag{1.47}$$

because

$$\delta \mathbf{r}_i = \sum_{j=1}^{m} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

for instantaneous displacements. From Eqs. (1.44) and (1.47) it follows that

$$\sum_{j=1}^{m} \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j^a \right) \delta q_j = 0 \tag{1.48}$$

The $\delta q_j$ which appear in Eq. (1.48) are not independent but must satisfy the instantaneous constraint equations

$$\sum_{j=1}^{m} A_{kj}\, \delta q_j = 0 \qquad k = 1, 2, \ldots, s \tag{1.49}$$

The "elimination" of $s$ of the $\delta q_j$ between Eqs. (1.48) and (1.49) is effected, in the usual way, by the introduction of $s$ Lagrange multipliers $\lambda_k (k = 1, 2, \ldots, s)$. This step leads directly to the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j^a - \sum_{k=1}^{s} \lambda_k A_{kj} \qquad j = 1, 2, \ldots, m \tag{1.50}$$

These $m$ second-order ordinary differential equations are the Lagrange equations of the system. The general solution of the equations is not available.* For a holonomic system with $n$ degrees of freedom, Eq. (1.50) reduces to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j^a \qquad j = 1, \ldots, n \tag{1.51}$$

In the presence of a function $V$ such that

$$Q_j^a = -\partial V/\partial q_j$$

and

$$\partial V/\partial \dot{q}_j = 0$$

Eqs. (1.51) can be written in the form

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_j} - \frac{\partial \mathscr{L}}{\partial q_j} = 0 \qquad j = 1, 2, \ldots, n \tag{1.52}$$

in which

$$\mathscr{L} = T - V$$

The scalar function $\mathscr{L}$—the lagrangian—which is the difference between the kinetic and potential energies is all that need be known to write the Lagrange equations in this case.

The major factor which contributes to the solving of Eq. (1.52) is the presence of ignorable coordinates. In fact, in dynamics problems, generally, the possibility of finding analytical representations of the motion depends on there being ignorable coordinates. A coordinate, say $q_k$, is said to be ignorable if it does not appear explicitly in the lagrangian, i.e., if

$$\partial \mathscr{L}/\partial q_k = 0 \tag{1.53}$$

If Eq. (1.53) is valid, then Eq. (1.52) leads to

$$\partial \mathscr{L}/\partial \dot{q}_k = \text{const} = c_k$$

and hence a first integral of the motion is available. Clearly the more ignorable coordinates that exist in the lagrangian, the better. This being so, considerable effort has been directed toward developing systematic means of generating ignorable coordinates by transforming from one set of generalized coordinates to another, more suitable, set. This transformation theory of dynamics, while extensively developed, is not generally of practical value in engineering problems.

### 1.5.3 The Euler Angles

To use lagrangian methods in analyzing the motion of a rigid body one must choose a set of generalized coordinates which uniquely determines the position of the body relative to a frame of reference fixed in space. It suffices to examine the motion of a body rotating about its center of mass.

An inertial set of orthogonal axes $\xi$, $\eta$, and $\zeta$ with origin at the center of mass and a noninertial set $x$, $y$, and $z$ fixed relative to the body with the same origin are adopted. The required generalized coordinates are those which specify the position of the $x$, $y$, and $z$ axes relative to the $\xi$, $\eta$, and $\zeta$ axes. More than one set of coordinates which achieves this purpose can be found. The most generally useful one, viz., the Euler angles, is used here.

*Nonholonomic problems are frequently more tractable by vectorial than by lagrangian methods.

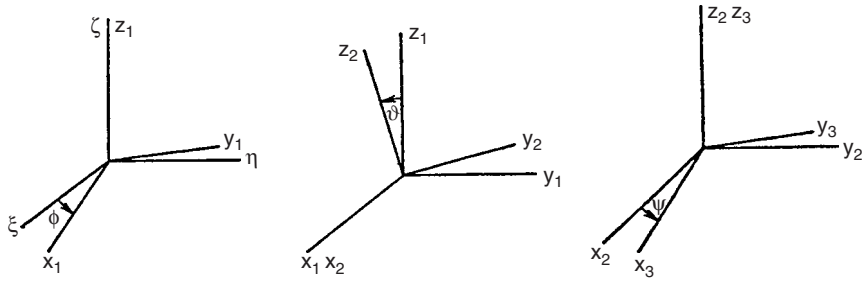


**FIG. 1.4**

The frame ξ, η, and ζ can be brought into coincidence with the frame *x, y*, and *z* by three finite rigid-body rotations through angles ϕ, θ, and ψ,* in that order, defined as follows (see Fig. 1.4):

1. A rotation about the ζ axis through an angle ϕ to produce the frame $x_1, y_1, z_1$
2. A rotation about the $x_1$ axis through an angle θ to produce the frame $x_2, y_2, z_2$
3. A rotation about the $z_2$ axis through an angle ψ to produce the frame $x_3, y_3, z_3$, which coincides with the frame *x, y, z*

Each rotation can be represented by an orthogonal matrix operation so that the process of getting from the inertial to the noninertial frame is

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{1.54a}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = B \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \tag{1.54b}$$

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = C \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \tag{1.54c}$$

Consequently,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = CBA \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = D \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{1.55}$$

where

$$D = CBA = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

*This notation is not universally adopted. See Ref. 5 for discussion.

Since $A$, $B$, and $C$ are orthogonal matrices, it follows from Eq. (1.55) that

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = D^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = D' \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.56}$$

where the prime denotes the transpose of the matrix. From Eq. (1.55) one sees that, if the time dependence of the three angles $\phi$, $\theta$, $\psi$ is known, the orientation of the $x$, $y$, $z$ and axes relative to the $\xi$, $\eta$, and $\zeta$ axes is determined. This time dependence is sought by attempting to solve the Lagrange equations.

The kinetic energy $T$ of the rotating body is found from Eq. (1.31) to be

$$2T = I_{xx}\omega_x^2 + I_{yy}\omega_x^2 + I_{zz}\omega_z^2 \tag{1.57}$$

in which the components of the angular velocity $\omega$ are provided by the matrix equation

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = CB \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = C \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \tag{1.58}$$

It is to be noted that if

$$I_{xx} \neq I_{yy} \neq I_{zz} \tag{1.59}$$

none of the angles is ignorable. Hence considerable difficulty is to be expected in attempting to solve the Lagrange equations if this inequality, Eq. (1.59), holds. A similar inference could be made on examining Eq. (1.30). The possibility of there being ignorable coordinates in the problem arises if the body has axial, or so-called kinetic, symmetry about (say) the $z$ axis. Then

$$I_{xx} = I_{yy} = I$$

and, from Eq. (1.57),

$$2T = I(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_{zz}(\dot{\phi}\cos\theta + \dot{\psi})^2 \tag{1.60}$$

The angles $\phi$ and $\psi$ do not occur in Eq. (1.60). Whether or not they are ignorable depends on the potential energy $V(\phi, \theta, \psi)$.

### 1.5.4 Small Oscillations of a System near Equilibrium

The Lagrange equations are particularly useful in examining the motion of a system near a position of equilibrium. Let the generalized coordinates $q_1, q_2, \ldots, q_n$—the explicit appearance of time being ruled out—represent the configuration of the system. It is not restrictive to assume the equilibrium position at

$$q_1 \text{ and } q_2 = \cdots = q_n = 0$$

and, since motion near this position is being considered, the $q_i$ and $\dot{q}_i$ may be taken to be small.

The potential energy can be expanded in a Taylor series about the equilibrium point in the form

$$V(q_1 \cdots q_n) = V(0) + \sum_{i=1}^{n} \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_i \sum_j \left(\frac{\partial^2 V}{\partial q_i\, \partial q_j}\right)_0 q_i\, q_j + \cdots \tag{1.61}$$

In Eq. (1.61) the first term can be neglected because it merely changes the potential energy by a constant and the second term vanishes because $\partial V/\partial q_i$ is zero at the equilibrium point. Thus, retaining only quadratic terms in $q_i$, one finds

$$V(q_1 \cdots q_n) = \frac{1}{2} \sum_i \sum_j V_{ij}\, q_i\, q_j \tag{1.62}$$

in which

$$V_{ij} = (\partial^2 V/\partial q_i\, \partial q_j)_0 = V_{ji} \tag{1.63}$$

are real constants.

The kinetic energy $T$ of the system is representable by an analogous Taylor series

$$T(\dot{q}_i \cdots \dot{q}) = \frac{1}{2} \sum_i \sum_j T_{ij}\, \dot{q}_i\, \dot{q}_j \tag{1.64}$$

where

$$T_{ij} = T_{ji} \tag{1.65}$$

are real constants. The quadratic forms, Eqs. (1.62) and (1.64), in matrix notation, a prime denoting transposition are

$$V = \frac{1}{2} q' \mathcal{V} q \tag{1.66}$$

and

$$T = \frac{1}{2} \dot{q}' \mathcal{T} \dot{q} \tag{1.67}$$

In these expressions $\mathcal{V}$ and $\mathcal{T}$ represent the matrices with elements $V_{ij}$ and $T_{ij}$, respectively, and $q$ represents the column vector $(q_1, \ldots, q_n)$. The form of Eq. (1.67) is necessarily positive definite because of the nature of kinetic energy. Rather than create the Lagrange equations in terms of the coordinates $q_i$, a new set of generalized coordinages $\rho_i$ is introduced in terms of which the energies are simultaneously expressible as quadratic forms without cross-product terms. That the transformation to such coordinates is possible can be seen by considering the equations

$$\mathcal{V} b_j = \lambda_j \mathcal{T} b_j \qquad j = 1, 2, \ldots, n \tag{1.68}$$

in which $\lambda_j$, the roots of the equation

$$|\mathcal{V} - \lambda \mathcal{T}| = 0$$

are the eigenvalues—assumed distinct—and $b_j$ are the corresponding eigenvectors. The matrix of eigenvectors $b_j$ is symbolized by $B$, and the diagonal matrix of eigenvalues $\lambda_j$ by $\Lambda$. One can write

$$b_k' \mathcal{V} b_j = \lambda_j b_k' \mathcal{T} b_j$$

and

$$b_k' \mathcal{V} b_j = \lambda_k b_k' \mathcal{T} b_j$$

because of the symmetry of $\mathcal{V}$ and $\mathcal{T}$. Thus, if $\lambda_j \neq \lambda_k$, it follows that

$$b_k' \mathcal{T} b_j = 0 \qquad k \neq j$$

and, since the eigenvectors of Eq. (1.68) are each undetermined to within an arbitrary multiplying constant, one can always normalize the vectors so that

$$b_i' \mathcal{T} b_i = 1 \tag{1.69}$$

Hence

$$B' \mathcal{T} B = \mathrm{I}$$

where I is the unit matrix. But

$$\mathcal{V}B = \mathcal{T}B\Lambda \tag{1.70}$$

and so

$$B'\mathcal{V}B = B'\mathcal{T}B\Lambda = \Lambda \tag{1.71}$$

Furthermore, denoting the complex conjugate by an overbar, one has

$$\mathcal{V}\bar{b}_j = \bar{\lambda}_j \mathcal{T}\bar{b}_j$$

and

$$b_j'\mathcal{V}\bar{b}_j = \bar{\lambda}_j b_j'\mathcal{T}\bar{b}_j \tag{1.72}$$

since $\mathcal{V}$ and $\mathcal{T}$ are real. However,

$$b_j'\mathcal{V}\bar{b}_j = \lambda_j b_j'\mathcal{T}\bar{b}_j \tag{1.73}$$

because $\mathcal{V}$ and $\mathcal{T}$ are symmetric. From Eqs. (1.72) and (1.73) it follows that

$$(\lambda_j - \bar{\lambda}_j)b_j'\mathcal{T}\bar{b}_j = 0 \tag{1.74}$$

The symmetry and positive definiteness of $\mathcal{T}$ ensure that the form $b_j'\mathcal{T}\bar{b}_j$ is real and positive definite. Consequently the eigenvalues $\lambda_j$, and eigenvectors $b_j$, are real. Finally, one can solve Eq. (1.68) for the eigenvalues in the form

$$\lambda_j = b_j'\mathcal{V}b_j / b_j'\mathcal{T}b_j \tag{1.75}$$

The transformation from the $q_i$ to the $\rho_i$ coordinates can now be made by writing

$$q = B\rho$$

from which

$$V = \frac{1}{2}q'\mathcal{V}q = \frac{1}{2}\rho'B'\mathcal{V}B\rho = \frac{1}{2}\rho'\Lambda\rho \tag{1.76}$$

and

$$T = \frac{1}{2}\dot{q}'\mathcal{T}\dot{q} = \frac{1}{2}\dot{\rho}'B'\mathcal{T}B\dot{\rho} = \frac{1}{2}\dot{\rho}'\mathrm{I}\dot{\rho} \tag{1.77}$$

It is seen from Eqs. (1.76) and (1.77) that $V$ and $T$ have the desired forms and that the corresponding Lagrange equations (1.52) are

$$d^2\rho_i/dt^2 + \omega_i^2\rho_i = 0 \qquad i = 1, \ldots, n \tag{1.78}$$

where $\omega_i^2 = \lambda_i$. If the equilibrium position about which the motion takes place is stable, the $\omega_i^2$ are positive. The eigenvalues $\lambda_i$ must then be positive, and Eq. (1.75) shows that $V$ is positive definite. In other words, the potential energy is a minimum at a position of stable equilibrium. In this case, the motion of the system can be analyzed in terms of its normal modes—the $n$ harmonic oscillators Eq. (1.78). If the matrix $V$ is not positive definite, Eq. (1.75) indicates that negative eigenvalues may exist, and hence Eqs. (1.78) may have hyperbolic solutions. The equilibrium is then unstable. Regardless of the nature of the equilibrium, the Lagrange equations (1.78) can always be arrived at, because it is possible to diagonalize simultaneously two quadratic forms, one of which (the kinetic-energy matrix) is positive definite.

### 1.5.5 Hamilton's Principle

In conclusion it is remarked that the Lagrange equations of motion can be arrived at by methods other than that presented above. The point of departure adopted here is Hamilton's principle, the statement of which for holonomic systems is as follows.

Provided the initial ($t_1$) and final ($t_2$) configurations are prescribed, the motion of the system from time $t_1$ to time $t_2$ occurs in such a way that the line integral

$$\int_{t_1}^{t_2} \mathcal{L}\, dt = \text{extremum}$$

where $\mathcal{L} = T - V$. That the Lagrange equations [Eq. (1.52)] can be derived from this principle is shown here for the case of a single-mass, one-degree-of-freedom system. The generalization of the proof to include an $n$-degree-of-freedom system is made without difficulty.

The lagrangian is

$$\mathcal{L}(q, \dot{q}, t) = T - V$$

in which $q$ is the generalized coordinate and $q(t)$ describes the motion that actually occurs. Any other motion can be represented by

$$\bar{q}(t) = q(t) + \varepsilon f(t) \tag{1.79}$$

in which $f(t)$ is an arbitrary differentiable function such that $f(t_1)$ and $f(t_2) = 0$ and $\varepsilon$ is a parameter defining the family of curves $\bar{q}(t)$. The condition

$$\int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, t)\, dt = \text{extremum}$$

is tantamount to

$$\frac{\partial}{\partial \varepsilon} \int_{t_1}^{t_2} (\bar{q}_1, \dot{\bar{q}}_1, t)\, dt = 0 \qquad \varepsilon = 0 \tag{1.80}$$

for all $f(t)$. But

$$\frac{\partial}{\partial \varepsilon} \int_{t_1}^{t_2} \mathcal{L}(\bar{q}_1, \dot{\bar{q}}_1, t)\, dt = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \bar{q}} \frac{\partial \bar{q}}{\partial \varepsilon} + \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}} \frac{\partial \dot{\bar{q}}}{\partial \varepsilon} \right) dt$$

which, by Eq. (1.79), is

$$\frac{\partial}{\partial \varepsilon} \int_{t_1}^{t_2} (\bar{q}_1, \dot{\bar{q}}_1, t)\, dt = \int_{t_1}^{t_2} \left[ f(t) \frac{\partial \mathcal{L}}{\partial \bar{q}} + \dot{f}(t) \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}} \right] dt \tag{1.81}$$

Its second term having been integrated by parts, Eq. (1.81) reduces to

$$\frac{\partial}{\partial \varepsilon} \int_{t_1}^{t_2} \mathcal{L}(\bar{q}, \dot{\bar{q}}, t)\, dt = \int_{t_1}^{t_2} f(t) \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}} \right) dt$$

because $f(t_1) = f(t_2) = 0$. Hence Eq. (1.80) is equivalent to

$$\int_{t_1}^{t_2} f(t) \left( \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) dt = 0 \tag{1.82}$$

for all $f(t)$. Equation (1.82) can hold for all $f(t)$ only if

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

which is the Lagrange equation of the system.

The extension to an $n$-degree-of-freedom system is made by employing $n$ arbitrary differentiable functions $f_k(t)$, $k = 1, \ldots, n$ such that $f_k(t_1) = f_k(t_2) = 0$. For the generalizations of Hamilton's principle which are necessary in treating nonholonomic systems, the references should be consulted.

The principle can be extended to include continuous systems, potential energies other than mechanical, and dissipative sources. The analytical development of these and other topics and examples of their applications are presented in Refs. 4 and 8 through 12.

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